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Phase-Coupled Nonlinear Dynamical Systems, Stability, First Integrals, and Liapunov Exponents

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The stability of a class of coupled identical autonomous systems of first-order nonlinear ordinary differential equations is investigated. These couplings play a central role in controlling chaotic systems and can be applied in electronic circuits and laser systems. As applications we consider a coupled van der Pol equation and a coupled logistic map. When the uncoupled system admits a first integral we study whether a first integral exists for the coupled system. Gradient systems and the Painlevé property are also discussed. Finally, the relation of the Liapunov exponents of the uncoupled and coupled systems are discussed.

Consider the autonomous system of first-order ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$. We assume that the functions F_j: $\mathbf{R}^n \to \mathbf{R}$ are analytic. Assume that $\mathbf{u}^*$ is a fixed point of (1), i.e., $\mathbf{F}(\mathbf{u}^*) = \mathbf{0}$. The variational equation of (1) is given by (Steeb, 1992, 1993, 1994)

$$\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}}(\mathbf{u}(t))\mathbf{y} \tag{2}$$

where $\partial\mathbf{F}/\partial\mathbf{u}$ is the Jacobian matrix. Inserting the fixed point $\mathbf{u}^*$ into the Jacobian matrix results in an $n \times n$ matrix

$$A := \frac{\partial \mathbf{F}}{\partial \mathbf{u}}(\mathbf{u} = \mathbf{u}^*) \tag{3}$$

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with constant coefficients. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of this matrix determine the stability of the fixed point $\mathbf{u}^*$. Furthermore the eigenvalues provide information as to whether Hopf bifurcation can occur. In this case we assume that $\mathbf{F}$ depends on a (bifurcation) parameter. Moreover, the variational system (2) is used to find the one-dimensional Liapunov exponents.

In controlling the chaos of the autonomous system (1) ($n \geq 3$) the coupling of two identical systems according to

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) + c(\mathbf{v} - \mathbf{u}), \qquad \frac{d\mathbf{v}}{dt} = \mathbf{F}(\mathbf{v}) \tag{4}$$

and

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) + c(\mathbf{v} - \mathbf{u}), \qquad \frac{d\mathbf{v}}{dt} = \mathbf{F}(\mathbf{v}) + c(\mathbf{u} - \mathbf{v}) \tag{5}$$

plays a central role (Steeb *et al.*, 1995, Fujisaka and Yamada, 1983; van Wyk and Steeb, 1997). Here $c \in \mathbf{R}$. First we realize that $(\mathbf{u}^*, \mathbf{v}^*)$ with $\mathbf{v}^* = \mathbf{u}^*$ is a fixed point of (4) and (5) if $\mathbf{u}^*$ is a fixed point of (1). Inserting the fixed point $(\mathbf{u}^*, \mathbf{u}^*)$ into the Jacobian matrix associated with (4) and (5) yields a $2n \times 2n$ matrix. We now show that the eigenvalues of this $2n \times 2n$ matrix can be found from the eigenvalues of the $n \times n$ matrix given by A. Then from the $2n$ eigenvalues of (2) we can determine the stability of the fixed point $(\mathbf{u}^*, \mathbf{u}^*)$ for the systems (4) and (5).

First we consider system (4). To find the eigenvalues we prove the following.

Theorem 1. Let A be an $n \times n$ matrix over the real numbers. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Define the $2n \times 2n$ matrix M as

$$M := \begin{pmatrix} A - cI_n & cI_n \\ 0_n & A \end{pmatrix} \tag{6}$$

where I_n is the $n \times n$ unit matrix. Then the eigenvalues of M are given by

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \quad \lambda_1 - c, \lambda_2 - c, \ldots, \lambda_n - c \tag{7}$$

Proof. There exists an $n \times n$ orthogonal matrix Q such that $Q^T A Q = D + U$, where $D := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and U is a strictly upper-triangular $n \times n$ matrix. Let

$$P := \begin{pmatrix} Q & 0_n \\ 0_n & Q \end{pmatrix} \tag{8}$$

where O_n is the $n \times n$ zero matrix. Thus

$$P^{-1} = \begin{pmatrix} Q^T & 0_n \\ 0_n & Q^T \end{pmatrix} \tag{9}$$

It follows that

$$P^{-1}MP = \begin{pmatrix} (D - cI_n) + U & cI_n \\ 0_n & D + U \end{pmatrix} \tag{10}$$

This is an upper-triangular matrix. The entries on the diagonal are $\lambda_1, \ldots, \lambda_n, \lambda_1 - c, \ldots, \lambda_n - c$, which are the eigenvalues of M. This proves the theorem.

Obviously the matrix M is the matrix which follows from the Jacobian matrix of system (4) after inserting the fixed point $(\mathbf{u}^*, \mathbf{u}^*)$.

Next we consider the coupled system (5). To find these eigenvalues we prove the following.

Theorem 2. Let A be an $n \times n$ matrix over the real numbers. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Define the $2n \times 2n$ matrix M as

$$M := \begin{pmatrix} A - cI_n & cI_n \\ cI_n & A - cI_n \end{pmatrix} \tag{11}$$

where I_n is the $n \times n$ unit matrix. Then the eigenvalues of M are given by

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \quad \lambda_1 - 2c, \lambda_2 - 2c, \ldots, \lambda_n - 2c \tag{12}$$

Proof. There exists an $n \times n$ orthogonal matrix Q such that $Q^TAQ = D + U$, where $D := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and U is a strictly upper-triangular $n \times n$ matrix. Let

$$P := \begin{pmatrix} Q & 0_n \\ Q & Q \end{pmatrix} \tag{13}$$

where 0_n is the $n \times n$ zero matrix. Thus

$$P^{-1} = \begin{pmatrix} Q^T & 0_n \\ -Q^T & Q^T \end{pmatrix} \tag{14}$$

It follows that

$$P^{-1}MP = \begin{pmatrix} D + U & cI_n \\ 0_n & (D - 2cI_n) + U \end{pmatrix} \tag{15}$$

This is an upper-triangular matrix. The entries on the diagonal are $\lambda_1, \ldots, \lambda_n, \lambda_1 - 2c, \ldots, \lambda_n - 2c$, which are the eigenvalues of M. This proves the theorem.

Obviously the matrix M is the matrix which follows from the Jacobian matrix of system (4) after inserting the fixed point $(\mathbf{u}^*, \mathbf{u}^*)$.

As an application, let us consider the van der Pol equation

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = r(1 - u_1^2)u_2 - u_1 \tag{16}$$

Then $\mathbf{u}^* = (0,0)$ is a fixed point of (16). The eigenvalues of the functional matrix for this fixed point are given by $\lambda_{1,2} = r/2 \pm (r^2/4 - 1)^{1/2}$. Thus the uncoupled system shows Hopf bifurcation (Steeb, 1993). We find Hopf bifurcation when r crosses the imaginary axis. For the van der Pol equation a stable limit cycle is born. If we consider the coupling due to systems (5), we find that the eigenvalues of the coupled system are given by $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$, $\mu_3 = \lambda_1 - 2c$, $\mu_4 = \lambda_2 - 2c$.

Theorems 1 and 2 can also be applied to coupled maps. For example, Theorem 2 can be applied to

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) + c(\mathbf{y}_n - \mathbf{x}_n), \qquad \mathbf{y}_{n+1} = \mathbf{f}(\mathbf{y}_n) + c(\mathbf{x}_n - \mathbf{y}_n) \tag{17}$$

As an example, consider the logistic map with $f(x) = rx(1 - x)$. The map f admits the fixed points $x_1^* = 0$ and $x_2^* = (r - 1)/r$. In the following we consider the fixed point x_2^*. Then we find that the Jacobian (which is a 1×1 matrix) at this fixed point is given by $A = \lambda = 2 - r$. Thus the eigenvalues for the coupled system are $2 - r$, $2 - r - 2c$. An electronic circuit for the phase-coupled logistic map $x_{t+1} = 1 - ax_t^2 - b(x_t - y_t)$, $y_{t+1} = 1 - ax_t^2 - b(y_t - x_t)$ has been described by Mishina *et al.* (1985).

Let us now study first integrals of the system (1). Assume that (1) admits a first integral of the form $g(\mathbf{u})\exp(\epsilon t)$. Such first integrals appear in dissipative systems (Steeb, 1982; Steeb and Erig, 1983; Steeb and Euler, 1988). When $\epsilon = 0$ the nondissipative case is also included.

Theorem 3. Assume that (1) admits the first integral $g(\mathbf{u})\exp(\epsilon t)$. Then

$$(g(\mathbf{u}) + g(\mathbf{v}))\exp(\epsilon t) \tag{18}$$

is a first integral of the coupled system if

$$\frac{\partial g(\mathbf{u})}{\partial u_j} - \frac{\partial g(\mathbf{v})}{\partial u_j} = 0, \qquad j = 1, 2, \ldots, n \tag{19}$$

Proof. From the condition that $g(\mathbf{u})\exp(\epsilon t)$ is a first integral of (1) we find

$$\sum_{j=1}^{n} \frac{\partial g(\mathbf{u})}{\partial u_j} F_j(\mathbf{u}) + \epsilon g(\mathbf{u}) = 0 \tag{20}$$

Inserting (20) and (5) into

$$\frac{d}{dt}([g(\mathbf{u}) + g(\mathbf{v})]\exp(\epsilon t)) = 0 \tag{21}$$

we obtain

$$c\sum_{j=1}^{n}\left(\frac{\partial g(\mathbf{u})}{\partial u_j} - \frac{\partial g(\mathbf{v})}{\partial v_j}\right)(v_j - u_j) = 0 \tag{22}$$

This proves the theorem.

An example where we can apply this theorem is the Lotka–Volterra model,

$$\frac{du_1}{dt} = cu_1 + u_1(u_2 + u_3)$$

$$\frac{du_2}{dt} = cu_2 + u_2(u_3 - u_1)$$

$$\frac{du_3}{dt} = cu_3 + u_3(-u_1 - u_2) \tag{23}$$

An explicitly time-dependent first integral is given by $I(\mathbf{u}, t) = (u_1 + u_2 + u_3)\, e^{-ct}$.

Assume now that the system (1) is a gradient system, i.e., $\mathbf{F}(\mathbf{u}) = -\text{grad } W(\mathbf{u})$, where W is the potential. What can be said about the coupled system (5)? It is obvious that the coupled system (5) is also a gradient system, since it can be derived from the potential $W(\mathbf{u}) + W(\mathbf{v}) + \frac{1}{2}c(\mathbf{u} - \mathbf{v})^T(\mathbf{u} - \mathbf{v})$.

Let us now discuss the Painlevé test. Let $\mathbf{u}(t) = \mathbf{\Phi}(\mathbf{u}_0)$ be a solution of the initial value problem of (1). Let $\mathbf{v}(t) = \mathbf{\Phi}_t(\mathbf{v}_0)$. Then $(\mathbf{u}(t), \mathbf{v}(t))$ is a particular solution of the coupled system (5) if $\mathbf{u}_0 = \mathbf{v}_0$. If $\mathbf{u}_0 \neq \mathbf{v}_0$ then $(\mathbf{u}(t), \mathbf{v}(t))$ is no longer a solution for (5). A similar argument can be applied to the Painlevé test Steeb and Euler (1988). If the uncoupled system (1) passes the Painlevé test, i.e., has an expansion (considered in the complex time domain) of

$$u_j(t) = (t - t_1)^{n_j}\sum_{j=0}^{\infty} a_j(t - t_1)^j \qquad (j = 1, 2, \ldots, n)$$

with the right number of Kowalevski exponents [see Steeb and Euler (1988) for more details], then the coupled system (5) admits an expansion of the form $(\mathbf{u}(t), \mathbf{v}(t))$ around the singularity t_1, but the number of free parameters is one less than required. Thus the coupled system does not pass the Painlevé test in general if the uncoupled system (1) passes the test.

Let us now consider the Liapunov exponents. To find a relation between the Liapunov exponents of the coupled system (5) and the uncoupled system (1) we consider the time evolution of $\boldsymbol{\Theta}(t) := \mathbf{u}(t) - \mathbf{v}(t)$. We call $\boldsymbol{\Theta}$ the phase difference (Fujisaka and Yamada, 1983). It follows that

$$\frac{d\boldsymbol{\Theta}}{dt} = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt} = \mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v}) - 2c\boldsymbol{\Theta} \tag{24}$$

where we used (5). Using a Taylor expansion for $\mathbf{F}(\mathbf{u})$ and $\mathbf{F}(\mathbf{v})$ and the fact that

$$\frac{\partial \mathbf{F}(\mathbf{u} = \mathbf{u}(t))}{\partial \mathbf{u}} = \frac{\partial \mathbf{F}(\mathbf{v} = \mathbf{u}(t))}{\partial \mathbf{v}} \tag{25}$$

we obtain

$$\frac{d\boldsymbol{\Theta}}{dt} = (A(t) - 2cI)\boldsymbol{\Theta} + O(\boldsymbol{\Theta}^2) \tag{26}$$

where $A(t) := \partial \mathbf{F}(\mathbf{u} = \mathbf{u}(t))/\partial \mathbf{u}$ and $O(\boldsymbol{\Theta}^2)$ indicate higher order terms in $\boldsymbol{\Theta}$. Integrating (26) and neglecting the higher order terms yields

$$\boldsymbol{\Theta}(t) = \left[e^{-2ct} T \exp\left(\int_0^t A(s)\, ds \right) \right] \boldsymbol{\Theta}(0) \tag{27}$$

where T is the time-ordering operator. The eigenvalues μ_j ($j = 1, 2, \ldots, n$) of

$$\lim_{t \to \infty} \left[T \exp\left(\int_0^t ds\, A(s) \right) \right] \tag{28}$$

are related to the Liapunov exponents λ_j ($j = 1, 2, \ldots, n$) of system (1) via $\lambda_j = \ln|\mu_j|$. We find

$$\langle |\boldsymbol{\Theta}(t)| \rangle \propto \exp[(\lambda_{\max} - 2c)t] \tag{29}$$

where the average is taken over all initial conditions $\mathbf{u}(0)$ and all directions of $\boldsymbol{\Theta}(0)$ and $\lambda_{\max}$ is the largest one-dimensional Liapunov exponent. Equation (29) tells us that for $2c > \lambda_{\max}$ both systems stay in phase. Consequently, they have the $\lambda_{\max}$ of the uncoupled system (1). The two systems get out of phase at the value $c^* = \lambda_{\max}/2$. Thus c^* provides the largest one-dimensional Liapunov exponent.

Thus one of the main applications of phase-coupled chaotic systems would be for controlling chaos in electronic and laser systems.

REFERENCES

Fujisaka, H., and Yamada, T. (1983). *Progress of Theoretical Physics*, **69**, 32.

Mishina, T., Kohomoto, T., and Hashi, T. (1985). *American Journal of Physics*, **53**, 332.

Steeb, W.-H. (1982). *Journal of Physics A: Mathematical and General*, **15**, L389.
Steeb, W.-H. (1992). *A Handbook of Terms Used in Chaos and Quantum Chaos*, BI-Wissenschaftsverlag, Mannheim.
Steeb, W.-H. (1993). *Chaos and Fractals, Algorithms and Computations*, Spektrum, Heidelberg.
Steeb, W.-H. (1994). *Chaos and Quanten Chaos in Dynamischen Systemen*, Spektrum, Heidelberg.
Steeb, W.-H., and Erig, W. (1983). *Lettre al Nuovo Cimento*, **36**, 188.
Steeb, W.-H., and Euler, N. (1988). *Nonlinear Evolution Equations and Painlevé Test*, World Scientific, Singapore.
Steeb, W.-H., Stoop, R., Lai Choy Heng, and Villet, C. M. (1995). *South African Journal of Science*, **91**, 351.
van Wyk, M. A., and Steeb, W.-H. (1997). *Chaos in Electronics*, Kluwer, Dordrecht.